Transverse Nature of electromagnetic waves:-

Maxwell's four equations are given as follows

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ — ①, $\vec{\nabla}\cdot\vec{B} = 0$ — ② $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ — ③ and

$\vec{\nabla}\times\vec{B} = \mu_0(\vec{J} + \vec{J_d})$ — ④

However $\vec{J} = \sigma\vec{E}$ and $\vec{J_d} = \frac{\partial\vec{D}}{\partial t}$ where $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$\therefore \vec{J_d} = \frac{\partial}{\partial t}(\epsilon_0\vec{E} + \vec{P}) = \epsilon_0\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t}$

Put these values in ④, we get

$\vec{\nabla}\times\vec{B} = \mu_0\left(\sigma\vec{E} + \epsilon_0\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t}\right)$ — ⑤

But in vacuum, we have $\sigma = 0$, $\rho = 0$ and $\vec{P} = 0$ Therefore, Maxwell's four equations simplify to following form

$\vec{\nabla}\cdot\vec{E} = 0$ — ⑥, $\vec{\nabla}\cdot\vec{B} = 0$ — ⑦, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ — ⑧ and $\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ — ⑨

We know that solution of Maxwell's electromagnetic wave equation in vacuum for electric and magnetic field components are given as follows:-

$$\vec{E} = \vec{E_0}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \text{ and } \vec{B} = \vec{B_0}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \quad — ⑩$$

Where $\vec{E_0}, \vec{B_0}$ are amplitudes of Electric and magnetic fields

$\vec{E}, \vec{B}$ are instantaneous values of Electric and magnetic fields

$\omega = 2\pi\nu$ = angular frequency

$\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$ = propagation vector along the direction of propagation of electromagnetic wave.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ = position vector of any point where $\vec{E}$ & $\vec{B}$ are defined in equations ⑩.

Thus $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$ — ⑪

The solutions given in equation ⑩ must simultaneously obey equations ⑥ — ⑨ i.e. Maxwell's equations. Only then ⑩ will be true solutions of Maxwell's wave equation.

Let E_x, E_y, E_z are components of electric field. TN (2)

From (10), we can write ~~Eox~~ $E_x = E_{0x}\, e^{i(\omega t - \vec{k}\cdot\vec{r})}$

$$\left.\begin{aligned} E_x &= E_{0x}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \\ E_y &= E_{0y}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \\ E_z &= E_{0z}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \end{aligned}\right\} \quad \text{--- (12)}$$

Similarly components of magnetic field can be expressed as:-

$$\left.\begin{aligned} B_x &= B_{0x}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \\ B_y &= B_{0y}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \\ \text{and } B_z &= B_{0z}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \end{aligned}\right\} \quad \text{--- (13)}$$

~~Let us assert that~~ $\vec{E}$ will obey equation (6) if

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\text{or} \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \text{--- (14)}$$

From (12), we have

$$\frac{\partial}{\partial x} E_x = \left[E_{0x}\, e^{i(\omega t - \vec{k}\cdot\vec{r})}\right] \times \frac{d}{dx}\left[i(\omega t - \vec{k}\cdot\vec{r})\right]$$

$$= E_{0x}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} \times \cancel{(i\omega t - k_x)}\,(i)(-k_x)$$

$$= -i k_x E_{0x}\, e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

$$= -i k_x E_x \quad \text{--- (15)}$$

$$\text{Similarly} \quad \frac{\partial}{\partial y} E_y = -i k_y E_y \quad \text{--- (16)}$$

$$\text{and} \quad \frac{\partial}{\partial z} E_z = -i k_z E_z \quad \text{--- (17)}$$

Put values from (15), (16) and (17) in (14), we get

$$-i(k_x E_x + k_y E_y + k_z E_z) = 0$$

$$\Rightarrow \; -i(\vec{k}\cdot\vec{E}) = 0$$

$$\Rightarrow \; \vec{k}\cdot\vec{E} = 0$$

or $\vec{E} \perp \vec{k}$

Thus electric field is $\perp$ to direction of propagation.

Similarly we can show that $\vec{B}$ can satisfy equation 7 only if $\vec{B} \perp \vec{k}$ (i.e. $\vec{k}\cdot\vec{B} = 0$)

Thus magnetic field is also $\perp$ to direction of propagation.

To find relative orientation between $\vec{E}$ & $\vec{B}$, we must validate equations 8 and 9 by the solutions 16 of wave equation.

Now we can write

$$\vec{\nabla}\times\vec{E} = \hat{i}\,(\vec{\nabla}\times\vec{E})_x + \hat{j}\,(\vec{\nabla}\times\vec{E})_y + \hat{k}\,(\vec{\nabla}\times\vec{E})_z \quad \text{— (18)}$$

where $(\vec{\nabla}\times\vec{E})_x$ etc. are various components of $\vec{\nabla}\times\vec{E}$.

$$\text{Now } (\vec{\nabla}\times\vec{E})_x = \frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y \quad \text{— (19)}$$

$$\begin{array}{c|ccc} \hat{i} & \frac{\partial}{\partial x} & E_x \\ \hat{j} & \frac{\partial}{\partial y} & E_y \\ \hat{k} & \frac{\partial}{\partial z} & E_z \end{array}$$

From 12 we have

$$\frac{\partial}{\partial y}E_z = \left[E_{0z}\, e^{i(\omega t - \vec{k}\cdot\vec{r})}\right] \times \frac{\partial}{\partial y}\left[i(\omega t - \vec{k}\cdot\vec{r})\right]$$

$$= \left[E_{0z}\, e^{i(\omega t - \vec{k}\cdot\vec{r})}\right](i)(-k_y)$$

$$= -i k_y E_z$$

Similarly $\frac{\partial}{\partial z}E_y = -i k_z E_y$

Put in 19, we get $(\vec{\nabla}\times\vec{E})_x = -i\left[k_y E_z - k_z E_y\right]$

$$= -i(\vec{k}\times\vec{E})_x \quad \text{— (20)}$$

Similarly one can prove that

$$(\vec{\nabla}\times\vec{E})_y = -i(\vec{k}\times\vec{E})_y \quad \text{— (21)}$$

$$\text{and } (\vec{\nabla}\times\vec{E})_z = -i(\vec{k}\times\vec{E})_z \quad \text{— (22)}$$

Put values from 20, 21 and 22 in 18, we get

$$\vec{\nabla}\times\vec{E} = -i\left[\hat{i}(\vec{k}\times\vec{E})_x + \hat{j}(\vec{k}\times\vec{E})_y + \hat{k}(\vec{k}\times\vec{E})_z\right]$$

$$= -i(\vec{k}\times\vec{E}) \quad \text{— } 23$$

Again from 12, we can show that

$$\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\vec{B}_0\, e^{i(\omega t-\vec{k}\cdot\vec{r})}$$

$$= \left[\vec{B}_0\, e^{i(\omega t-\vec{k}\cdot\vec{r})}\right]\times\frac{\partial}{\partial t}\left[i(\omega t-\vec{k}\cdot\vec{r})\right]$$

$$= i\omega\left[\vec{B}_0\, e^{i(\omega t-\vec{k}\cdot\vec{r})}\right]$$

$$= i\omega\vec{B} \quad \text{— } 24$$

Put values from 23 and 24 in 8, we get

$$-i(\vec{k}\times\vec{E}) = -i\omega\vec{B}$$

$$\text{or}\quad \boxed{\vec{B} = \frac{1}{\omega}(\vec{k}\times\vec{E})} \quad \text{— } 25$$

Equation 25 shows that $\vec{B}$ is $\perp$ to $\vec{E}$

Thus from validation of first three equations, we get that $\vec{E}$, $\vec{B}$ & $\vec{k}$ are mutually $\perp$ to each other. Hence electromagnetic waves are transverse in nature.

<u>Note</u> We can also satisfy equation 9 although it is not neccessary now. However, it can come as additional problem in exam.

We can see that $\vec{\nabla}\times\vec{B} = -i(\vec{k}\times\vec{B})$ — 26 (HW)

$$\text{and}\quad \frac{\partial\vec{E}}{\partial t} = i\omega\vec{E} \quad \text{— } 27 \quad (HW)$$

Put these values in 9, we get

$$-i(\vec{k}\times\vec{B}) = \mu_0\epsilon_0(i\omega\vec{E})$$

$$\therefore\quad \boxed{\vec{E} = \frac{-1}{\omega\mu_0\epsilon_0}(\vec{k}\times\vec{B})} \quad \text{— } 28$$

This equation also tells that $\vec{E}$ is $\perp$ to $\vec{B}$.