Subject : Mathematics I
Sub. code : BSC - 103

MST - 2

Solution + Marking Scheme

Q1 (a)

$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$

$$AX = B$$

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{array}\right]$$

$R_2 - 3R_1$, $R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda+18 & 9 \end{array}\right]$$

$R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda+5 & 0 \end{array}\right]$$

$1\frac{1}{2}$

Case I If $\lambda \neq -5$

$\rho(A) = 3$ and $\rho(A:B) = 3$

$\rho(A) = \rho(A:B) = 3 =$ no. of unknowns.

unique solution.

$x + 2y - 3z = -2$

$-7y + 13z = 9$

$(\lambda+5)z = 0$

$z = 0$, $y = -9/7$, $x = 4/7$.

(1½)

Case II If $\lambda = -5$

$\rho(A) = 2$ $\rho(A:B) = 2$

$\rho(A) = \rho(A:B) = 2 <$ no. of unknowns

Infinitely many solutions.

$x + 2y - 3z = -2$

$-7y + 13z = 9$

Let $y = k$, $z = \frac{9+7k}{13}$, $x = \frac{1-5k}{13}$

(1)

(b) $\sum_{n=1}^{\infty} n e^{-n^2}$

$f(x) = x e^{-x^2}$

$f(x)$ is non-negative $\forall x \geq 1$

$$f'(x) = \frac{e^{x^2}(1) - x e^{x^2} 2x}{(e^{x^2})^2}$$

$$= \frac{e^{x^2}(1-2x^2)}{(e^{x^2})^2} < 0 \quad \forall \; x \geq 1$$ (1½)

$f(x)$ is monotonically decreasing

Since $f(x)$ is non negative and monotonically decreasing for $x \geq 1$

Cauchy's Integral Test is applicable.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x}{e^{x^2}}\,dx$$

Put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$= \int_1^\infty \frac{dt}{2e^t} = \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_1^\infty$$

$$= \frac{1}{2e} \text{ finite}$$ (2)

$\int_1^\infty f(x)\,dx$ cgs by cauchy integral test Σa_n also converge. (½)

Q 2 (a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^3}$

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n+1}{n^3} \right| = \sum \frac{n+1}{n^3}$$

(1)

$a_n = \frac{n+1}{n^3}$ Let $b_n = 1/n^2$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^3} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}(1+1/n)}{\cancel{n^3}} = 1$$

(2)

$\sum \frac{1}{n^2}$ is convergent by p-series test

$\sum a_n$ is also cgt by limit comparision test.

The given series converges absolutely.

(b) $\frac{1^2}{4^2} + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \cdots$

$$a_n = \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2}$$

(1/2)

$$a_{n+1} = \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2 (4n+4)^2}$$

$$\frac{a_n}{a_{n+1}} = \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2} \times \frac{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2 (4n+4)^2}{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2 (4n+1)^2}$$

$$\frac{a_n}{a_{n+1}} = \frac{(4n+4)^2}{(4n+1)^2}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(4n+4)^2}{(4n+1)^2}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

$1\frac{1}{2}$

Ratio Test fails

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{(4n+4)^2}{(4n+1)^2} - 1\right)$$

$$= n\frac{(24n+15)}{(4n+1)^2}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{\cancel{n^2}\left(24 + 15/n\right)}{\cancel{n^2}\left(4 + 1/n\right)^2} = \frac{24}{16} = \frac{3}{2} > 1$$

By Raabe's test, the given series converges.

$\sum_{n=1}^{\infty} a_n$ cgt.

2

Q3 $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$

$(\lambda+1)(\lambda+1)(\lambda+3) = 0$

Eigen values are $-1, -1, 3$

3

when $\lambda = 3$

$(A - 3I) X_1 = 0$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = -1$

3

$X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -1/2 \\ 2 & -1 & -1/2 \end{bmatrix}$$ 1

$$P^{-1}AP = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -1/2 \\ 2 & -1 & -1/2 \end{bmatrix} \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$ 1